Oblique derivative problems for elliptic and parabolic equations, Lecture I

Gary M. Lieberman

Iowa State University

July 20, 2011
1. Introduction
   - Basic Ideas
   - Outline of future lectures

2. Pointwise estimates
First, we present some terminology for parabolic equations. We use $X = (x, t)$ and $Y = (y, s)$ to denote points in $\mathbb{R}^{n+1}$, and we write $\Omega$ for a bounded open subset of $\mathbb{R}^n$. We use the usual parabolic norm:

$$|X| = \max\{|x|, |t|^{1/2}\}$$

and cylinders

$$Q(Y, r) = \{X : |X - Y| < r, t < s\}.$$
There are several parts of the boundary of $\Omega$ that will be important. First, we write $\mathcal{P}\Omega$ for the parabolic boundary, which consists of all points $Y$ such that any cylinder $Q(Y, r)$ contains points not in $\overline{\Omega}$. We also assume that there is a unique number $T_0$ such that if $Y \in \mathcal{P}\Omega$ and $Q(Y, r)$ contains no points of $\Omega$, then $s = t_0$. We write $B\Omega$ for the set of all $Y \in \mathcal{P}\Omega$ with $s = t_0$, and we write $S\Omega = \mathcal{P}\Omega \setminus B\Omega$. 
The oblique derivative problem for parabolic equations is easy to state. It’s

\[-u_t + a^{ij} D_{ij} u + b^i D_i u + cu = f \text{ in } \Omega,\]
\[\beta^i D_i u + \beta^0 u = g \text{ on } S\Omega,\]
\[u = \varphi \text{ on } B\Omega.\]

When \( P\Omega \) is sufficiently smooth, we write \( \gamma \) for the unit inner spatial normal to \( S\Omega \) and we assume that \( \beta \cdot \gamma > 0 \). We also assume that the matrix \( [a^{ij}] \) is positive-definite.
The easiest example is the Neumann problem for the heat equation on a cylindrical domain $\Omega = \omega \times (0, T)$:

$$-u_t + \Delta u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial \gamma} = g \text{ on } \partial \omega \times (0, T), \ u = \varphi \text{ on } \omega.$$ 

It’s well-known that, if this problem has a solution, then the solution is as smooth as the data allow. For example, if $\partial \omega \in H_{2+\alpha}$, $f \in H_{\alpha}(\Omega)$, $g \in H_{1+\alpha}(S\Omega)$, and $\varphi \in H_{2+\alpha}(\omega)$, then $u \in H_{2+\alpha}(\Omega)$. 
In these lectures, we will look at oblique derivative problems for linear equations and boundary conditions and also some nonlinear ones. Our main goal is to look at lots of a priori estimates, but these will be tools in proving existence of solutions. To start, we recall the Hölder spaces. For a bounded open subset $\Omega$ of $\mathbb{R}^n$ and a real number $\alpha \in (0, 1]$, we define the $\alpha$-Hölder seminorm of a function $u$ by the formula

$$[u]_{\alpha} = \sup \left\{ \frac{|u(X) - u(Y)|}{|X - Y|^\alpha} : X \neq Y \text{ in } \Omega \right\}.$$

We also write $|u|_0$ for the supremum of $u$ over $\Omega$, and we write

$$|u|_{\alpha} = |u|_0 + [u]_{\alpha}$$

for the Hölder norm of $u$. 
For parabolic equation, we also need a special additional seminorm. For $\beta \in (0, 2]$, we define

$$\langle u \rangle_\beta = \sup \left\{ \frac{|u(x, t) - u(x, s)|}{|t - s|^{\beta/2}} \right\},$$

where the supremum is over $(x, t)$ and $(x, s)$ in $\Omega$ with $s \neq t$. 
For $\alpha \in (0, 1]$, we further define

$$[u]_1 = |Du|_0 + \langle u \rangle_1, \quad [u]_2 = |D^2u|_0 + |u_t|_0 + \langle Du \rangle_1,$$

$$[u]_{1+\alpha} = [Du]_{\alpha} + \langle u \rangle_{1+\alpha},$$

$$[u]_{2+\alpha} = [D^2u]_{\alpha} + [u_t]_{\alpha} + \langle Du \rangle_{1+\alpha},$$

and $|u|_{k+\alpha} = |u|_0 + [u]_{k+\alpha}$ for $k = 0, 1, 2$. 
Sometimes, we will use weighted Hölder seminorms. For \( b \geq -\alpha \), we set

\[
[u]^{(b)}_\alpha = \sup \{ d(X)^{b+\alpha} \frac{|u(X) - u(Y)|}{|X - Y|^{\alpha}} : X \in \Omega, |X - Y| \leq \frac{1}{2} d(X) \},
\]

where

\[
d(X) = \sup \{|X - Y| : Y \in \partial \Omega, s \leq t\}
\]

is the parabolic distance from \( X \) to \( \partial \Omega \). We also define

\[
[u]^{(b)}_{1+\alpha} = [Du]^{(b+1)}_\alpha + \sup \{ d(X)^{b+1+\alpha} \frac{|u(x, t) - u(x, s)|}{|t - s|^{1+\alpha}} \}
\]

for \( b \geq -1 - \alpha \) and a similar definition for \([u]^{(b)}_{2+\alpha}\).
The weighted norms are
\[ |u|_a^{(b)} = |u|_0^{(b)} + [u]_a^{(b)}, \]
where
\[ |u|_0^{(b)} = \begin{cases} \sup (\text{diam } \Omega)^b |u| & \text{if } b \leq 0, \\ \sup d^b |u| & \text{if } b > 0. \end{cases} \]
As we have said, if the boundary data are sufficiently smooth, then there is no problem with making sense of the boundary condition

\[ \beta \cdot Du = g \]

with \( \beta \cdot \gamma > 0 \). (Recall that \( \gamma \) is the unit spatial inner normal to \( \partial \Omega \).) For several reasons, we’ll want to look at this condition when the boundary don’t have a normal everywhere, so we need to give a generalized meaning to both the equation \( \beta \cdot Du = g \) and the inequality \( \beta \cdot \gamma > 0 \).
We say that a nonzero vector $\beta_0$ is *inward pointing* at $X_0 \in S\Omega$ if there is a positive number $\tau_0$ such that, for all $\tau \in (0, \tau_0)$, $X_0 + \tau \beta_0 \in \bar{\Omega}$. A vector field $\beta$ is *inward pointing on* $S \subset S\Omega$ if $\beta(X_0)$ is inward pointing at $X_0$ for all $x_0 \in S$. We then say that $\beta \cdot Du \geq g$ at $X \in S\Omega$,

$$
\liminf_{\tau \to 0^+} \frac{u(X + \tau \beta(X)) - u(X)}{\tau} \geq g(X).
$$

with a similar definition for $\beta \cdot Du \leq g$. Hence $\beta \cdot Du = g$ means the directional derivative of $u$ in the direction $\beta(X)$ exists and equals $g(X)$. 
Except for the simplest results, we need a stronger version of the inward pointing condition. To define obliqueness, we assume that $\Omega$ satisfies an interior parabolic cone condition, which means that there are positive constants $R$, $\omega_0$ and $\omega_1$, and a vector $\gamma \in \mathbb{R}^n$ such that, if $X \in Q(X_0, R)$ satisfies
\[
\gamma \cdot (x - x_0) > \omega_0[|x - x_0|^2 - |\gamma \cdot (x - x_0)|^2]^{1/2} + \omega_1|t - t_0|^{1/2},
\]
then $X \in \Omega$. We say that $\beta$ is oblique at $X_0$ if the angle between $\beta$ and this vector $\gamma$ is less than $\arctan \omega_0$. 
We’ll start with the theory for linear problems. As for the Dirichlet problem, we want to prove, eventually, that solutions are in $H_{2+\alpha}(\overline{\Omega})$ under suitable hypotheses. But, in order to study nonlinear problems, we’ll look at intermediate estimates under weaker hypotheses.
First, we prove an estimate on $|u|_0$ for fairly general domains using only $L^{n+1}$ bounds on the coefficients $b^i$ and $c$, and $L^\infty$ bounds on $a^{ij}$, $\beta^i$, and $\beta^0$. These bounds will lead to a Harnack inequality and a Hölder estimate. Next, we prove a $H^{1+\alpha}$ estimate on $u$. The surprising part of this estimate is that we can do so for parabolically Lipschitz domains. Finally, we prove the $H^{2+\alpha}$ estimate in $C^{1,\alpha}$ domains.

We’ll do the linear theory for parabolic equations, and the nonlinear theory, mostly, for elliptic equations.
Our first estimate is in a general domain, but the hypotheses are a little unusual. Note that any oblique vector field points into $\Omega$ but also any vector field that is tangential to $S\Omega$ is inward pointing. Note that, if $u$ has a maximum at some $X \in S\Omega$ and if $\beta(X)$ is inward pointing at $X$, then $\beta \cdot Du(X) \leq 0$. 
For brevity, we define

\[ Lu = -u_t + a^{ij} D_{ij} u + b^i D_i u + cu, \quad Mu = \beta \cdot Du + \beta^0 u, \]

and we write \( b \) for the vector \((b^1, \ldots, b^n)\). We also introduce the upper contact set of \( u \), written \( E(u) \), which is the set of all points \( X \in \Omega \) such that there is a \( \xi \in \mathbb{R}^n \) such that

\[ u(Y) \leq u(X) + \xi \cdot (y - x) \quad (1) \]

for all \( Y \in \Omega \) with \( s \leq t \). For \( \theta \in (0, 1] \) and \( \beta_0 \) and \( R \) positive, we write \( E_\theta(u) \) for the subset of \( E(u) \) on which \( u \geq 0 \), and (1) holds with \( \xi \) also satisfying

\[ \frac{R + \beta_0}{\theta} |\xi| \leq u(X) - \xi \cdot x \leq \frac{1}{2} \sup u. \]
Here is our maximum principle:
Suppose \( b/\mathcal{D}^{1/(n+1)} \in L^{n+1}(\Omega) \), \( c \leq 0 \), and set

\[
B_0 = \| b/\mathcal{D}^{1/(n+1)} \|_{n+1}.
\]

Suppose also that are positive constant \( \beta_0 \) such that \( |\beta| \leq \beta_0 \)
and \( R \) such that \( |x| \leq R \) in \( \Omega \). Then there is a constant \( C \),
determined only by \( n \) such that \( Lu \geq f \) in \( \Omega \), \( \beta \cdot Du \geq u \) on
\( S\Omega \), and \( u \leq 0 \) on \( B\Omega \) imply that

\[
\max_{\Omega} u \leq C(B_0 + R + \beta_0)^{n/(n+1)} \| f/\mathcal{D}^{1/(n+1)} \|_{n+1}.
\]

When \( \beta_0 = 0 \), this has been proved by many authors.
The proof consists of a few simple pieces. First, we suppose that \( u \in C^2(\Omega) \), and we prove an estimate without using the differential equation.

Define \( \Phi : \Omega \to \mathbb{R}^{n+1} \) by \( \Phi(X) = (Du(X), u(X) - x \cdot Du(X)) \). Then \( \det(\hat{D}\Phi) = u_t \det D^2u \) (where \( \hat{D} \) denotes the derivative with respect to space and time), so

\[
\int_{E_\theta} |u_t \det D^2u| \, dX \geq |\Phi(E_\theta)|.
\]

We’ll show that we can estimate \( \sigma = \sup u \) in terms of \( |\Phi(E_\theta)| \).

Let \( X_0 \in \Omega \) so that \( u(X_0) = \sigma \), and set

\[
\Sigma = \{ \Xi = (\xi, h) : R + \frac{\beta_0}{\theta}|\xi| < h < \frac{\sigma}{2} \}.
\]

We’ll show that \( \Sigma \subset \Phi(E_\theta) \) and then calculate \( |\Sigma| \leq |\Phi(E_\theta)| \).
Fix $\Xi \in \Sigma$ and define $p$ by $p(Y) = \xi \cdot y + h$. Since $p > 0$ in $\Omega$, it follows that $p > u$ in $B\Omega$. In addition, $p(X_0) \leq R|\xi| + h < \sigma$, so $p(X_0) < u(X_0)$. Hence, there is a $Y_1 \in \overline{\Omega}$ such that $p(Y_1) = u(y_1)$ and $p(Y) > u(Y)$ if $Y \in \overline{\Omega}$ with $s < s_1$. If $Y_1$ were in $S\Omega$, we would have

$$\xi \cdot Y_1 + h = u(Y_1) \leq \beta \cdot Du(Y_1) \leq \beta \cdot Dp(Y_1) = \beta \cdot \xi.$$ 

Hence $-R|\xi| + h \leq \beta_0|\xi|$, which contradicts $h > (R + \beta_0)|\xi|/\theta$. Therefore $Y_1 \in \overline{\Omega} \setminus P\Omega$, and, for all $Y \in \Omega$ with $s \leq s_1$, we have

$$u(Y) \leq \xi \cdot y + h = \xi \cdot (y - y_1) + h + p(Y_1) = \xi \cdot (y - y_1) + u(Y_1),$$

so $Y_1 \in E_\theta$. 
It follows that $\Xi \in \Phi(E_\theta)$, which implies that

$$\int_{E_\theta} |u_t \det D^2 u| \, dX \geq |\Phi(\Xi)| = \frac{\omega_n \sigma^{n+1} \theta^n}{n(n+1)2^{n+1}(R + \beta_0)^n}.$$ 

Simple algebra yields

$$\sigma \leq C(n) \left( \frac{R + \beta_0}{\theta} \right)^{n/(n+1)} \left( \int_{E_\theta} |u_t \det D^2 u| \, dX \right)^{1/(n+1)}.$$ 

A simple approximation argument shows that this inequality is valid even if $u \in W_{n+1}^{2,1}$. 
Now we use the differential equation. It’s easy to check that $D^2 u \leq 0$ and $u_t \geq 0$ a.e. on $E_\theta$, so the matrix inequality $\text{tr}(AB) \geq c(n)[\det(A) \det(B)]^{1/(n+1)}$ (which holds for all positive semidefinite matrices $A$ and $B$) gives

$$|u_t \det(D^2 u)| \leq c(n) \frac{(u_t - a^{ij} D_{ij} u)^{n+1}}{D} \leq c(n) \frac{(|b| |Du| + |f|)^{n+1}}{D}.$$

Since $|Du| \leq \sigma \theta/(R + \beta_0)$ on $E_\theta$, it follows that

$$\sigma \leq C(n) \left( \frac{R + \beta_0}{\theta} \right) \|f/D^{1/(n+1)}\|_{n+1} + C(n) \left( \frac{\theta B_0}{R + \beta_0} \right)^{1/(n+1)} \sigma.$$

Choosing $\theta$ sufficiently small finishes the proof.
Since the only restrictions on $\beta$ are that it is inward pointing and bounded, this result applies to the Dirichlet problem (with $\beta_0 = 0$) but it is inadequate to handle the Neumann problem for the heat equation. Here’s how to modify the estimate for general oblique conditions but we must make an additional assumption on $S\Omega$, that $S\Omega \in H_1$, which means that there is a constant $R_0$ such that, for any $x_0 \in S\Omega$, there is an $H_1$ function (of $n - 1$ spatial variables and one time variable) so that $Q(X_0, R_0) \cap S\Omega$ is the graph of this function and $Q(X_0, R_0) \cap \Omega$ lies on one side of the graph.
For simplicity, we assume that we can write \( Q(X_0, R_0) \cap \Omega \) as the set where \( x^n > \omega(x', t) \), and then we define \( w(X_0; X) = x^n - \omega(x', t) \) with \( |D\omega| \leq \omega_0 \) for some positive constant \( \omega_0 \) such that

\[
\omega_0 |\beta'| \leq (1 - \varepsilon) \beta^n
\]

for some \( \varepsilon \in (0, 1) \). (This condition is the appropriate uniform obliqueness condition on \( \beta \).)
With $\eta > 0$ a positive constant to be chosen, we take $\hat{\omega}$ to be a $C^\infty$ function with $|D\hat{\omega}| \leq \omega_0$ and $|\omega - \hat{\omega}| < \eta$. Then we define $w(X_0; X) = x^n - \hat{\omega}(X')$. It follows that

$$\beta \cdot Dw \geq C(\varepsilon, \omega_0) |\beta|.$$ 

We now cover $S\Omega$ by finitely many such cylinders $Q(X_j, R_0)$ and write $w_j$ for $w(X_j; \cdot)$ converted to the original coordinates. If $(\psi_j)$ is a partition of unity subordinate to this cover, then $\rho^* = \sum \psi_j w_j$ satisfies

$$\beta \cdot D\rho^* \geq |\beta|(C(\omega_0, \varepsilon) - \eta \sum |D\psi_j|),$$

and we can make the expression in parentheses greater than $C/2$ by taking $\eta$ sufficiently small.
Now suppose that we have

$$Lu \geq f \text{ in } \Omega, \beta \cdot Du + \beta_0 u \geq 0 \text{ on } S\Omega, u \leq 0 \text{ on } B\Omega.$$  

with $\beta_0 \leq k_0|\beta|$ for some nonnegative constant $k_0$. We set

$$v = u \exp(k_1 \rho^*)$$

with $k_1$ a constant to be chosen. The important calculation is

$$\beta \cdot Dv = \exp(k_1 \rho^*)[\beta \cdot Du + (k_1)u\beta \cot D\rho^*]$$

$$\geq \exp(k_1 \rho^*)u[\beta^0 + k_1 \beta \cdot D\rho^*].$$

By taking $k_1$ sufficiently large (and assuming without loss of generality that $|\beta| = 1$, we obtain $\beta \cdot Dv \geq v$.  

A similar calculation shows that

\[-v_t + a^{ij} D_{ij} v + \tilde{b}^i D_i v + \tilde{c} v \geq \tilde{f}\]

in \(\Omega\) with \(\tilde{b}\) a known function (in terms of \(b, k_1, \) and \(\rho^*\)), \(\tilde{c} \leq k\) for some nonnegative constant \(k\), and \(\tilde{f} = \exp(k_1 \rho^*)\). Finally, we set \(w = \exp(kt)\) to see that \(w\) satisfies

\[-w_t + a^{ij} D_{ij} w + \tilde{b}^i D_i \tilde{c} w \geq \hat{f} \text{ in } \Omega,\]

\[\beta \cdot Dw \geq w \text{ on } S\Omega, w \leq 0 \text{ on } B\Omega\]

with \(\hat{c} = \tilde{c} - k \leq 0\). Applying our previous estimate to \(w\) and rewriting in terms of \(u\) gives

\[\sup u \leq C \|f / \mathcal{D}^{1/(n+1)}\|_{n+1}.\]
The preliminary estimate for elliptic problems is discussed, with corrections, in the second lecture.